Probing Teachers' Pedagogical Content Knowledge: Lessons from the Case of the Subtraction Algorithm

Helen L. Chick
University of Melbourne
h.chick@unimelb.edu.au

Thuy Pham
University of Melbourne
t.pham12@ugrad.unimelb.edu.au

Monica K. Baker
University of Melbourne
mbaker@huntingtower.vic.edu.au

In this paper we present a framework for investigating teachers' mathematical pedagogical content knowledge (PCK). The components of the framework permit identification of strengths and weaknesses in the PCK held by various teachers on different topics. It was applied to teachers' written and interview responses addressing a student's difficulties with the standard subtraction algorithm. This enabled comparison of the PCK held by individual teachers, together with analysis of the teachers' PCK as a group. Most understood the issues and knew suitable representations, but occasionally lacked key understanding of students' misconceptions and how to help students recognise them.

Investigating teachers' pedagogical content knowledge (PCK) is challenging for many reasons. PCK reveals itself in many places—in teachers' planning, classroom interactions, explanations, mathematical competency, and so on—and a study of only one environment will give a limited perspective. In addition, PCK's multifaceted nature makes considering all aspects time-consuming and complex. This report uses a framework that provides a set of lenses for studying PCK to address this second problem. At the same time we show that one carefully chosen environment can still reveal detail about teachers' PCK.

Background

A Framework for PCK

In the mid to late 1980s Shulman (1986, 1987) highlighted the complexity of teachers' knowledge by identifying several categories of knowledge important for teaching. Key aspects of this complexity are captured in his now widely-used term *pedagogical content knowledge*, which incorporates "an understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of learners, and presented for instruction" (Shulman, 1987, p. 8). Shulman and others have contributed to our understanding of particular aspects of PCK. For example, Shulman (1986) discussed the need for knowledge of student thinking, knowledge of models that illustrate key concepts, and understanding what contributes to the intrinsic difficulty of certain topics. Others (e.g., Graeber, 1999; Leinhardt, Putnam, Stein, & Baxter,1991) have stressed that teachers need knowledge of why confusions and misconceptions may occur. Ma (1999) recognised the centrality of content knowledge, but particularly highlighted the importance, for teaching, of a special kind of content knowledge, termed *Profound Understanding of Fundamental Mathematics* (PUFM). This, too, is implicit in Ball's focus on the ability to deconstruct concepts to make central ideas apparent (Ball, 2000); to go beyond the "what" to the "how" and "why". Ball also emphasised constructing appropriate explanations that are correct and suitable for the intended audience. Finally, making connections between topics and concepts, rather than presenting disparate components, has been shown to produce higher gains in student learning (Askew, Brown, Rhodes, Johnson, & Wiliam, 1997).

This literature (plus examples excluded due to space constraints) together with the authors' observations for a project investigating teachers' PCK, contributed to the development of the PCK framework shown in Figure 1 (from Chick, Baker, Pham, & Cheng, in press). This framework organises key elements of PCK into three categories. The first, "clearly PCK", is where pedagogy and content interact inseparably. This involves knowing about teaching strategies (e.g., suitable explanations and activities), student thinking (e.g., misconceptions, learning styles, levels of development), alternative models or representations, and resources and curriculum. The second category—"content knowledge in a pedagogical context"—focuses on mathematics content as used for teaching. Examples include deconstructing knowledge to its key components, highlighting mathematical connections, and demonstrating PUFM. The final category—"pedagogical knowledge in a content context"—considers generic teaching knowledge used in the specific case of mathematics teaching. It includes knowing how to keep students focused, identifying goals for teaching, and general classroom techniques such as grouping. The right-hand column of Figure 1 describes actions that provide evidence for each aspect of PCK.

The framework can be refined further by subdividing some elements. For example, a teacher might identify likely misconceptions about a topic (knowledge of general misconceptions in students' thinking) or identify that a certain student has a specific problem (knowledge of a specific student misconception). For data analysis this distinction between general and specific cases is useful, but we have combined these categories to save space. It is also acknowledged that there is overlap among categories (most noticeably in the elements concerned with content knowledge, where, for instance, capacity to deconstruct understanding also contributes to or provides evidence for PUFM). It is useful to keep them separate, however, as they can allow us to focus on deeper distinctions in PCK.

The framework has been used to study primary teachers' responses to a question about decimals (Chick, et al., in press). Although the item involved a narrow hypothetical situation (rather than, say, an actual lesson), use of the framework revealed a surprising amount of detail about teachers' PCK. This paper continues the study of the framework's effectiveness for examining PCK by looking at knowledge of the subtraction algorithm.

The Subtraction Algorithm and the Present Study

The "regrouping" or "decomposition" algorithm for subtraction is widespread in Australia and the US (e.g., Ross & Pratt-Cotter, 1999), perhaps preferred because it can be modelled with multibase arithmetic blocks (MAB or base-ten blocks). Fuson (1992) suggested that the difficulties children have with subtraction algorithms may be associated with incomplete place value understanding. One of the better known errors is that in vertically aligned problems students will subtract a smaller digit from a larger digit regardless of which is in the subtrahend. This is the basis for the item used in this study. Initial work by two of the authors (Chick & Baker, 2005) found that when teachers encountered a student presenting this error, they typically suggested that they would re-explain the algorithm, often with the support of concrete materials like MAB. However, the quality of explanations varied, suggesting differences in PCK that might be illuminated with the PCK framework.

PCK Category	Evident when the teacher …
Clearly PCK	
Teaching Strategies	Discusses or uses strategies or approaches for teaching a mathematical concept
Student Thinking	Discusses or addresses student ways of thinking about a concept or typical levels of understanding
Student Thinking - Misconceptions	Discusses or addresses student misconceptions about a concept
Explanations	Explains a topic, concept or procedure
Cognitive Demands of Task	Identifies aspects of the task that affect its complexity
Appropriate and Detailed Representations of Concepts	Describes or demonstrates ways to model or illustrate a concept (can include materials or diagrams)
Knowledge of Resources	Discusses/uses resources available to support teaching
Curriculum Knowledge	Discusses how topics fit into the curriculum
Purpose of Content Knowledge	Discusses reasons for content being included in the curriculum or how it might be used
Content Knowledge in a Pedagogical Context	
Profound Understanding of Fundamental Mathematics	Exhibits deep and thorough conceptual understanding of identified aspects of mathematics
Deconstructing Content to Key Components	Identifies critical mathematical components within a concept that are fundamental for understanding and applying that concept
Mathematical Structure and Connections	Makes connections between concepts and topics, including interdependence of concepts
Procedural Knowledge	Displays skills for solving mathematical problems (conceptual understanding need not be evident)
Methods of Solution	Demonstrates a method for solving a maths problem
Pedagogical Knowledge in a Content Context	
Goals for Learning	Describes a goal for students' learning (may or may not be related to specific mathematics content)
Getting and Maintaining Student Focus	Discusses strategies for engaging students
Classroom Techniques	Discusses generic classroom practices

Figure 1. Framework for analysing Pedagogical Content Knowledge (based on Chick, Baker, Pham & Cheng, in press).

With this background in mind, this paper seeks to answer the following questions:

1. What PCK is held by primary teachers for the subtraction algorithm?
2. Is the PCK framework useful and adequate for studying PCK?

Methodology

The present study arose out of a project investigating primary teachers' mathematical PCK by means of questionnaires, interviews, and lesson observations. The data here came from 14 Australian Grade 5/6 teachers who completed a questionnaire about aspects of teaching mathematics and undertook a follow-up interview with the third author (see also Chick & Baker, 2005). The questionnaire considered a variety of issues associated with teaching mathematics; the subtraction item shown in Figure 2 was used for this study.

You notice a student working on these subtraction problems: $\begin{array}{r} 438 \\ \underline{-172} \\ 346 \end{array} \qquad \begin{array}{r} 5819 \\ \underline{-2673} \\ 3266 \end{array}$ What would you do to help this student?

Figure 2. The subtraction item from the questionnaire.

Teachers completed the questionnaire in their own time and returned written responses to the researchers. Interview questions were then prepared for each teacher, often seeking additional explanation of the questionnaire response. Interview questions thus varied among teachers, but usually included "How would you convince the student of his/her mistake?" The questionnaire-then-interview methodology was used to allow teachers to make considered responses to the questionnaire without feeling pressured to answer "on the spot". The questionnaire also primed them on issues explored further in the interview.

The questionnaire and transcribed interview data were analysed using the framework. PCK categories were assigned to excerpts from the responses; in most cases this recorded the presence of the corresponding aspect of PCK, but in some cases significant absences or weaknesses were also noted. Data were sorted by PCK type and examined to investigate the teachers' PCK about the subtraction algorithm and students' associated understanding.

Results

Case Studies — Amy and Brian

We first present excerpts from the responses of two teachers, Amy and Brian, as they explained how they would address 438 – 172. These extracts show how elements from the PCK framework are revealed in the teachers' interview responses. Amy's explanation below shows not only familiarity with procedure but *conceptual fluency* (Chick, 2003) when discussing place value. She demonstrates PUFM, the capacity to deconstruct the key concepts, and knowledge of what may be difficult for student's thinking.

> We start from the ones and we take away two ones. We'll put them over down here. Now, we've got three tens, but we need to take away 7 tens, so what are we going to do? "I know, I reckon we could trade one of these things, one hundred, for some tens". Oh good, how many tens in a hundred? Hundred, so ten. So I would say OK, well if you give me that hundred, how many of these do you want? Hopefully they'll say ten [counts out ten]. I would probably show, just to make absolutely certain that they believed me, that I hadn't made it up, and that everyone in the group could see, that it really is the same amount, we've still got the same number, we've just got it in a different way [showing that ten tens sitting on one hundred is the same size]. So now how many tens have we got? Umm, thirteen. I'd say, good, now can we take away the seven, and they'd say, yeah, not a problem [counts out 7]. [She goes on to complete the subtraction with the hundreds.]

Brian too can identify that the problem lies with not being able to subtract 70 from 30, so his procedural knowledge is correct. He seems to think, however, that the *student's* problem lies in not realising this. In fact, it is likely the student *does* recognise that 30-70 cannot be done, but resolves it by doing the subtraction that *can* be done. The problem is that the student does not know what to do when subtraction appears impossible.

> [If] they've got 438 – 172 a lot of children will go "Ooh, that's in the hundreds," […] but if you said what's eight minus two, and treat it as one subtraction, just with a simple digit, then they'd be able to handle that. […] If you did four minus one the same way [referring to the hundreds], then 30 minus 70, if you just put the 30-70 on the board they'd be able to tell you you can't do that. So

> taking it out of the context of the whole sum and breaking it down to smaller sections and treating as four separate things and then just explaining to them, here it all is in the one package in the one sum, and you've got to work your way systematically through it. […] If I wrote 30 – 70 on the board they would say, we can't do that, whereas if they see it in here, sometimes they either we do it 70 – 30, maybe it's meant to be, and they'll go that way. […]

Brian places no emphasis on place value or regrouping in his response (note the way he deals with the hundreds digits without referring to their place value), suggesting limited deconstruction of key underlying concepts. He has a useful teaching/diagnostic strategy in having students explain their thought processes. Elsewhere he also discussed the strategy of using other students' correct explanations to help students recognise their own difficulties. However, he could not suggest ways of getting students to realise that they *were* actually making a mistake, implying limited knowledge of structure and connections; Amy, in contrast, spontaneously mentioned using addition as a checking mechanism.

The framework thus allows us to identify the presence—and absence—of specific aspects of PCK. It has to be acknowledged, though, that we can only examine the PCK evident in what teachers said, and it may be that they have additional understanding that did not emerge during the questionnaire and interview process. This means we must be cautious in making comparisons among teachers. Nevertheless, the questions asked were designed to elicit PCK and since what teachers choose to emphasise also reflects their PCK observed differences may reflect significant underlying variations in levels of knowledge.

Specific Areas of PCK

We now look at the different aspects of PCK apparent among the teachers as a group, to ascertain the prevalence of desirable PCK associated with the subtraction algorithm. Note that the questionnaire-and-interview approach may favour certain categories of PCK over others that might be more evident in different situations.

Teaching strategies. About half the teachers discussed the importance of concrete representations as a foundation for understanding, with two specifically saying they would gradually reduce students' reliance on concrete materials and two suggesting that these materials would only be used for students who were really struggling. Another suggested strategy had students verbalising their thought processes while solving the problem, so the student might self-diagnose the error or reveal the point of difficulty to the teacher. One teacher would ask the more confident students to explain their procedure to the class before calling on students who were experiencing some difficulty.

Most of the teachers said they would re-explain the procedure used in the algorithm, often with MAB as the supporting concrete representation. In re-teaching, there was emphasis on regrouping and renaming, on showing that 7 cannot be subtracted from 3, on separating out components of the problem as three separate subtractions (although this was not clearly explained by the teachers due to some shortcomings in PUFM and their capacity to deconstruct the key components), on aligning the numbers in the written algorithm, and on using simpler examples. Two teachers reinforced concepts and encouraged students by placing ticks under the correct aspects of the students' answers.

Knowledge of student thinking. Several teachers identified strengths and weaknesses in their own students' performance on this topic: one said place value was a huge problem at the 5/6 level; another said lower ability students may be able to do the procedure but may not have conceptual understanding; a third stated that she had a bright class but needed to modify work for three or four weaker students; another chose from a variety of different methods depending on the students' level of understanding; and one admitted that sometimes students still did not understand even after reteaching. One teacher suggested students might experience tension in alternating between work with concrete materials and pen-and-paper techniques, adding that students may not have understood the whole concept in their initial work with concrete materials.

Student thinking — misconceptions. All the teachers appeared to identify the student error, but not all articulated the misconception clearly. Six teachers recognised the tendency for students to always subtract the smaller number from the larger number. Four teachers explicitly suggested that this was due to difficulty with regrouping. To counter the "subtract the smaller from the larger" approach, one teacher emphasised always working from top to bottom. To ascertain where students might be having difficulty, five of the teachers suggested asking students to demonstrate and articulate their method for solving the subtraction problem. Another suggested asking students to tackle simpler problems in order to locate students' difficulties. Finally one teacher mentioned giving a diagnostic pre-test before starting a topic as a general strategy to address misconceptions.

Explanations, procedural knowledge, and m*ethods of solution.* All fourteen teachers demonstrated the ability to apply the subtraction algorithm and describe a method for explaining it to children. Indeed, most teachers emphasised re-explanation as the best way to address students' difficulties. Various approaches were noted, although only a couple of teachers offered two alternatives (reteaching the regrouping process and checking by adding). Six teachers emphasised the order in which the digits were to be addressed (i.e., top to bottom, right to left), with one teacher drawing an arrow at the top of each number as a reminder for students. Five teachers stated that they would initially solve the problem using concrete materials (such as MAB) then show the representation on paper. Three teachers would reteach the borrowing strategy to students; two others mentioned breaking the problem down to deal with each place value in turn, but without being clear about the regrouping issue.

Cognitive demand. Nine teachers identified aspects affecting the cognitive demand for students. Five of these discussed starting with problems that do not involve regrouping, with three explicitly stating that regrouping caused the difficulty and one linking this to problems with place value. Another said the size of the numbers affected the complexity, and suggested re-teaching with two-digit examples before tackling three-digit problems.

Appropriate and detailed representations of concepts and *knowledge of resources.* Eleven teachers described how they would represent the numbers in the subtraction to facilitate students' conceptual understanding of place value, often using concrete materials such as MAB (mentioned by nine teachers, although not all of them clearly expressed how it could be used to model regrouping), general (but unspecified) base 10 materials, and counters (to show that it is not possible to subtract 7 from 3). Several teachers explicitly mentioned the written recording process of the algorithm and the need for annotations when regrouping, and three teachers mentioned using calculators as a resource to convince students of the solution to the problem.

Curriculum knowledge. Five teachers alluded to subtraction's place in the curriculum, often associated with their experience and expectations of their own students' thinking.

One claimed to have no students with difficulties in Grade 5, whereas two others noted the need to spend time on the topic even in Grade 6. One teacher said her students saw themselves as too old for MAB, but indicated that she sanctioned its use. A fifth teacher claimed that Grade 5/6 students lack competence with place value and regrouping because earlier maths classes did not emphasise conceptual understanding. She felt that a new curriculum focus on place value in the early years would reduce difficulties in future.

Profound understanding of fundamental mathematics, deconstructing content to key components, and m*athematical structure and connections.* Most teachers approached the question procedurally, without making connections to other concepts or showing profound understanding, perhaps because of the nature of the stimulus question itself. Nevertheless, most discussed place value fluently. Ten teachers identified critical mathematical components that facilitate students' understanding of the subtraction algorithm: eight highlighted regrouping, and six focused on place value. Only six teachers discussed the use of checking mechanisms to help students: three demonstrated knowledge of mathematical connections in suggesting the use of addition as a checking strategy, three allowed their students to use a calculator, one had students check the answer with MAB, and another encouraged students to estimate the answer first. Four teachers, when explicitly asked about how they would convince a student that the answer was wrong, could not come up with a strategy for doing so apart from re-explaining the correct procedure.

Goals for learning, getting and maintaining student focus, and c*lassroom techniques.* Most teachers described specific achievements that they were trying to accomplish in students' learning of subtraction. This included knowledge of place value and when to trade/regroup; others emphasised setting out of written work. One teacher encouraged students to estimate their answers first to develop number sense and another stated that one of her goals was for students to always check their answers. The nature of the hypothetical situation meant that teachers' thinking about student focus was rarely observed. One teacher described a trading game using two dice to create numbers of tens and units for subtraction. Five of the fourteen teachers spontaneously mentioned that they would organise their class in groups for specific teaching, usually by ability level (implying knowledge of student thinking as well). Three of the teachers said that the composition of their groups varied by topic and need.

Discussion and Conclusion

In general, the teachers seemed to have solid knowledge of the subtraction algorithm, of its place in the curriculum and in their own students' understanding, and of how to teach it (including both general teaching strategies, such as using concrete materials and grouping students, and approaches specific to the topic, such as explaining regrouping with MAB). There were, however, two notable problems. As suggested by the data and also reported in Chick and Baker (2005) most teachers responded to the request to "help" the student by re-explaining the algorithmic procedure, sometimes with the aid of appropriate materials. Fewer had strategies for helping students actually recognise the existence of a problem and thus appreciate the need to learn the method correctly. Secondly, whereas all teachers seemed to identify the student's misconception, some attributed it to not knowing that a larger number cannot be subtracted from a smaller, rather than not knowing *what to do* in this situation. Those who recognised the latter problem were successful in discussing place value and regrouping as critical to the use of the algorithm; those who did not were unlikely to discuss these key concepts.

The questionnaire and interview approach seemed to work well for gathering data about a topic common to the group of teachers, although individual teachers still varied in what they chose to emphasise. This implies that direct comparisons among teachers must come with caveats. The PCK framework allowed us to examine the data in detail for evidence for the extent of the teachers' knowledge. On this particular item all responses that appeared to exhibit PCK could be coded using at least one category of the framework. If this applies with other data, including data from actual lessons and post-lesson interviews, then this suggests that the framework may be approaching an exhaustive list. The overlap among categories appears unavoidable, but allows us to focus on critical distinctions among types of knowledge and may actually ensure that all PCK is observed.

Acknowledgments

This research was supported by Australian Research Council grant DP0344229. We thank the teachers for their kind involvement in the research.

References

Askew, M., Brown, M., Rhodes, V., Johnson, D., & Wiliam, D. (1997). *Effective teachers of numeracy. Final report*. London: King's College.

Ball, D. L. (2000). Bridging practices: Intertwining content and pedagogy in teaching and learning to teach. *Journal of Teacher Education, 51*(3), 241-247.

Chick, H. L. (2003). Pre-service teachers' explanations of two mathematical concepts. *Proceedings of the 2003 annual conference of the Australian Association for Research in Education* (Auckland, NZ, 29 Nov. — 3 Dec., 2003). From: http://www.aare.edu.au/03pap/chi03413.pdf

Chick, H. L., & Baker, M. (2005). Investigating teachers' responses to student misconceptions. In H. L. Chick & J. L. Vincent (Eds.) *Proceedings of the 29th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 249-256). Melbourne: PME.

Chick, H. L., Baker, M., Pham, T., & Cheng, H. (in press). Aspects of teachers' pedagogical content knowledge for decimals. In *Proceedings of the 30th annual conference of the International Group for the Psychology of Mathematics Education*.

Fuson, K. C. (1992). Research on whole number addition and subtraction. In D. A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 243-275). New York: Macmillan.

Graeber, A. O. (1999). Forms of knowing mathematics: What preservice teachers should learn. *Educational Studies in Mathematics*, *38*, 189-208.

Leinhardt, G., Putnam, R. T., Stein, M. K., & Baxter, J. (1991). Where subject knowledge matters. In J. Brophy (Ed.), *Advances in research on teaching: Teachers' knowledge of subject matter as it relates to their teaching practice* (Vol. 2, pp. 87-113). Greenwich, Connecticut: Jai Press, Inc.

Ma, L. (1999). *Knowing and Teaching Elementary Mathematics: Teachers' Understanding of Fundamental Mathematics in China and the United States*. Mahwah, NJ: Lawrence Erlbaum Associates.

Ross, S. C., & Pratt-Cotter, M. (1999). Subtraction from a historical perspective. *School Science and Mathematics, 99,* 389-393.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), 4-14.

Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review, 57*(1), 1-22.